Lecture

Discrete-Time
Signals and Systems

1. Discrete-Time Signals and Systems

- signal classification -> signals to be applied in digital filter theory within our course,
- some elementary discrete-time signals,
- discrete-time systems: definition, basic properties review, discrete-time system classification, input-output model of discrete-time systems -> system to be applied in digital filter theory within our course,
- Linear discrete-time time-invariant system description in time-, frequency- and transform-domain.

1.1. Basic Definitions

1.1.1. Discrete and Digital Signals
1.1.1.1. Basic Definitions

Signals may be classified into four categories depending on the characteristics of **the time-variable** and **values** they can take:

- continuous-time signals (analogue signals),
- discrete-time signals,
- continuous-valued signals,
- discrete-valued signals.

Continuous-time (analogue) signals:

Time: defined for every value of time $t \in R$,

Descriptions: functions of a continuous variable t: $f(t)$,

Notes: they take on values in the continuous interval $f(t) \in (-a,b)$ $for\ a,b \to \infty$.

Note: $f(t) \in C$

$$f(t) = \sigma + j\omega$$

$$\sigma \in (-a,b)\, and\, \omega \in (-a,b)$$

$$a,b \to \infty$$

Discrete-time signals:

Time: defined **only** at discrete values of time: $t = nT$,

Descriptions: sequences of real or complex numbers $f(nT) = f(n)$,

Note A.: they take on values in the continuous interval $f(n) \in (-a, b)$ $for\ a, b \to \infty$,

Note B.: sampling of analogue signals:

- sampling interval, period: T,
- sampling rate: *number of samples per second*,
- sampling frequency (Hz): $f_S = 1/T$.

Continuous-valued signals:

Time: they are defined **for every value of time** or only at discrete values of time,

Value: they can take on **all possible values** on finite or infinite range,

Descriptions: functions of a continuous variable or sequences of numbers.

Discrete-valued signals:

Time: they are defined for **every value of time** or only at discrete values of time,

Value: they can take on values from **a finite set** of possible values,

Descriptions: functions of a continuous variable or sequences of numbers.

Digital filter theory:

Discrete-time signals:

Definition and descriptions: defined only at **discrete** values of time and they can take **all** possible values on finite or infinite range (**sequences** of real or complex numbers: $f(n)$),

Note: sampling process, constant sampling period.

Digital signals:

Definition and descriptions: **discrete-time and discrete-valued signals** (i.e. discrete -time signals taking on values from a finite set of possible values),

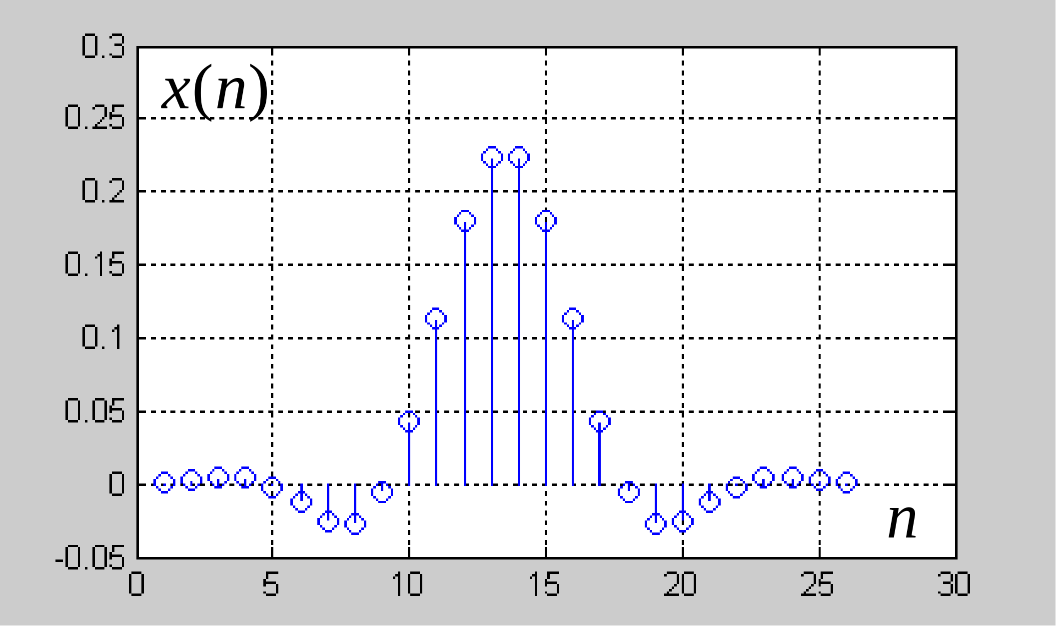
Note: sampling, quatizing and coding process i.e. process of analogue-to-digital conversion.

1.1.1.2. Discrete-Time Signal Representations

A. Functional representation:

$$x(n)=\begin{cases}1 & for \quad n=1,3\\ 6 & for \quad n=0,7\\ 0 & elsewhere\end{cases} \qquad y(n)=\begin{cases}0 & for \quad n<0\\ 0{,}6^{n} & for \quad n=0,1,\ldots,102\\ 1 & \quad n>102\end{cases}$$

B. Graphical representation



C. Tabular representation:

n	…	-2	-1	0	1	2
$x(n)$	…	0.12	2.01	1.78	5.23	0.12

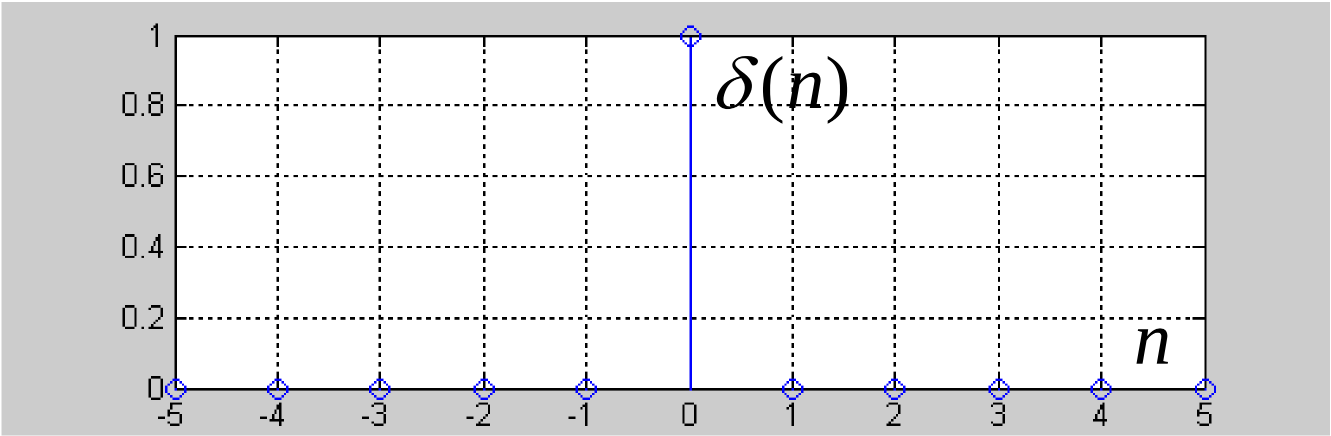
D. Sequence representation:

$$x(n) = \{\ldots \quad 0.12 \quad 2.01 \quad 1.78 \quad 5.23 \quad 0.12 \quad \ldots\}$$

1.1.1.3. Elementary Discrete-Time Signals

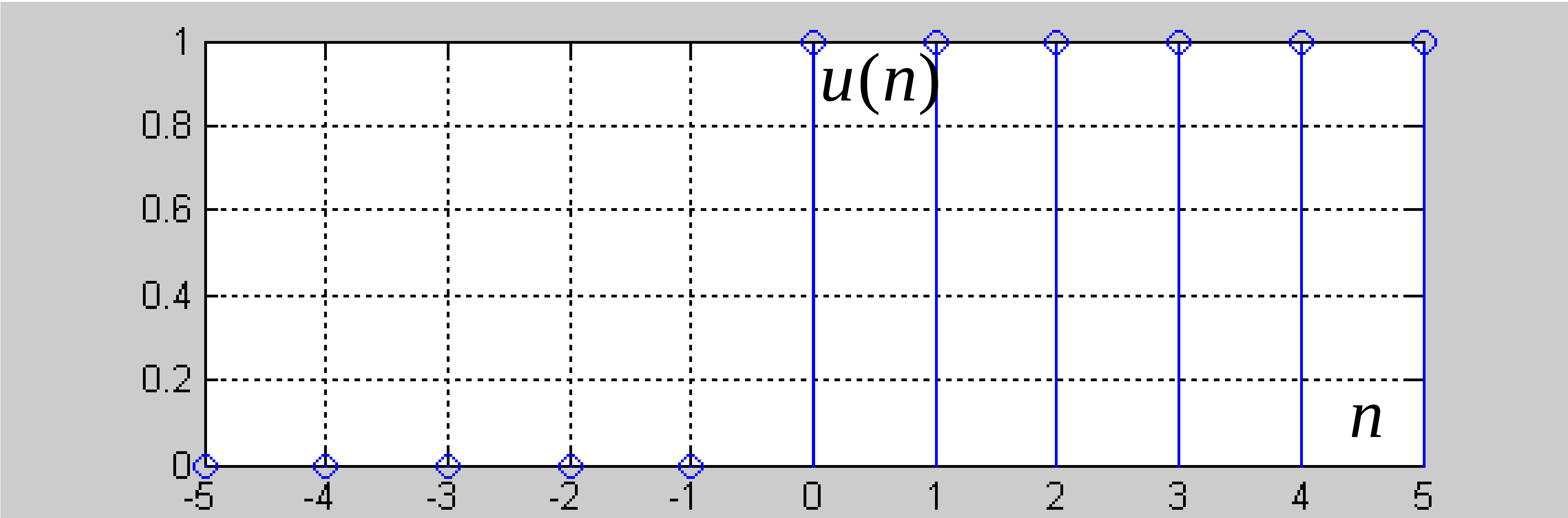
A. Unit sample sequence (unit sample, unit impulse, unit impulse signal)

$$\delta(n) = \begin{cases} 1 & for\ n = 0 \\ 0 & for\ n \neq 0 \end{cases}$$



B. Unit step signal (unit step, Heaviside step sequence)

$$u(n) = \begin{cases} 1 & for \quad n \geq 0 \\ 0 & for \quad n < 0 \end{cases}$$



C. Complex-valued exponential signal

(complex sinusoidal sequence, complex phasor)

$$x(n) = e^{j\omega nT}, \ |x(n)| = 1, \ \arg[x(n)] = \omega nT = 2\pi f.nT = \frac{2\pi f.n}{f_S}$$

where

$$\omega \in R, \ n \in N, \ j = \sqrt{-1} \ is \ imaginary \ unit$$

and

T is sampling period and f_S is sampling frequency.

1.1.2. Discrete-Time Systems. Definition

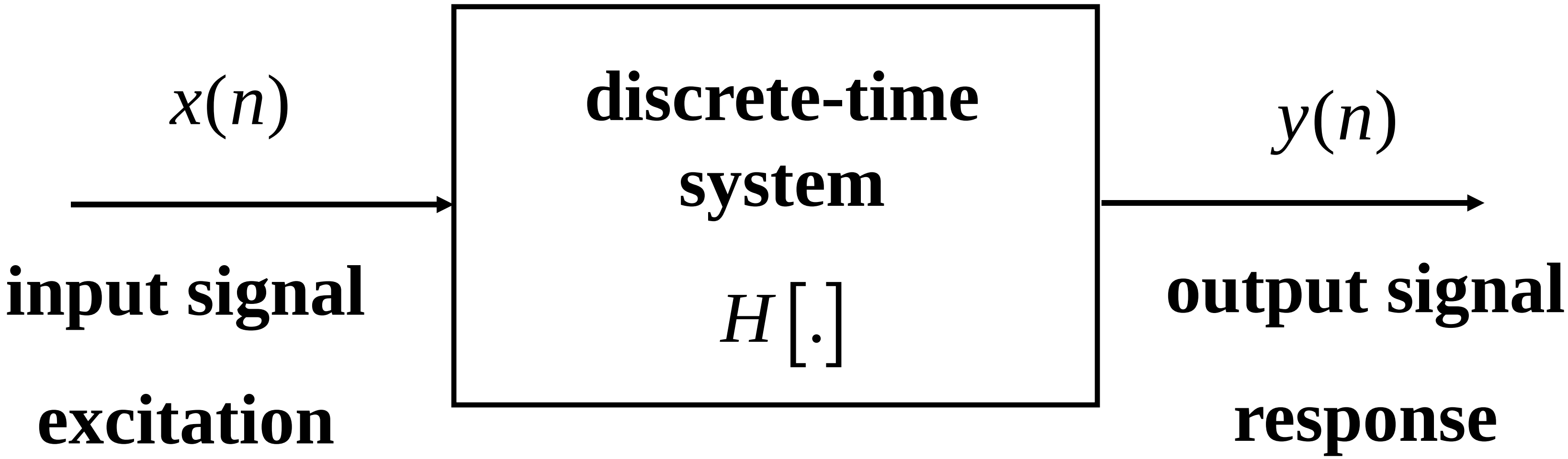
A discrete-time system is a device or algorithm that operates on a discrete-time signal called ***the input*** or ***excitation*** (e.g. ***x(n)***), according ***to some rule*** (e.g. ***H[.]***) to produce another discrete-time signal called *the* ***output*** or ***response*** (e.g. ***y(n)***).

$$y(n) \equiv H\left[x(n)\right]$$

This expression denotes also **the transformation *H[.]* (also called operator or mapping) or processing performed by the system on *x(n)* to produce *y(n)*.**

Input-Output Model of Discrete-Time System

(input-output relationship description)



$$y(n) \equiv H\left[x(n)\right]$$

$$x(n) \xrightarrow{H} y(n)$$

1.1.3. Classification of Discrete-Time Systems

1.1.3.1. Static vs. Dynamic Systems. Definition

A discrete-time system is called ***static*** or ***memoryless*** if its output at any time instant *n* depends on the input sample at the same time, but not on the past or future samples of the input. In the other case, the system is said to be ***dynamic*** or to have ***memory***.

If the output of a system at time *n* is completly determined by the input samples in the interval from *n*-*N* to *n* ($N \geq 0$), the system is said to have memory of ***duration N***.

If $N = 0$, the system is ***static*** or ***memoryless***.

If $0 < N < \infty$, the system is said to have ***finite memory***.

If $N \to \infty$, the system is said to have ***infinite memory***.

Examples:

The static (memoryless) systems:

$$y(n) = nx(n) + bx^3(n)$$

The dynamic systems with finite memory:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

The dynamic system with infinite memory:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

1.1.3.2. Time-Invariant vs. Time-Variable Systems. Definition

A discrete-time system is called ***time-invariant*** if its input-output characteristics do not change with time. In the other case, the system is called ***time-variable***.

Definition. A relaxed system $H[.]$ is ***time-*** or ***shift-invariant*** if only if

$$y(n) \equiv H\left[x(n)\right] \qquad x(n) \xrightarrow{H} y(n)$$

implies that

$$y(n-k) \equiv H\left[x(n-k)\right] \qquad x(n-k) \xrightarrow{H} y(n-k)$$

for ***every input signal*** $x(n)$ and ***every time shift*** k .

Examples:

The time-invariant systems:

$$y(n) = x(n) + bx^3(n)$$

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

The time-variable systems:

$$y(n) = nx(n) + bx^3(n-1)$$

$$y(n) = \sum_{k=0}^{N} h^{N-n}(k)x(n-k)$$

1.1.3.3. Linear vs. Non-linear Systems. Definition

A discrete-time system is called ***linear*** if only if it satisfies the ***linear superposition principle***. In the other case, the system is called ***non-linear***.

Definition. A relaxed system $H[.]$ is ***linear*** if only if

$$H\left[a_1x_1(n)+a_2x_2(n)\right]=a_1H\left[x_1(n)\right]+a_2H\left[x_2(n)\right]$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$, and any arbitrary constants a_1 and a_2.

Examples:

The linear systems:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) \qquad y(n) = x(n^2) + bx(n-k)$$

The non-linear systems:

$$y(n) = nx(n) + bx^3(n-1) \quad y(n) = \sum_{k=0}^{N} h(k)x(n-k)x(n-k+1)$$

1.1.3.4. Causal vs. Non-causal Systems. Definition

Definition. A system is said to be ***causal*** if the output of the system at any time n (i.e., $y(n)$) depends only on present and past inputs (i.e., $x(n)$, $x(n-1)$, $x(n-2)$, ...). In mathematical terms, the output of a ***causal*** system satisfies an equation of the form

$$y(n) = F\left[x(n), x(n-1), x(n-2), \cdots\right]$$

where $F[.]$ is some arbitrary function. If a system does not satisfy this definition, it is called ***non-causal***.

Examples:

The causal system:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) \qquad y(n) = x^2(n) + bx(n-k)$$

The non-causal system:

$$y(n) = nx(n+1) + bx^3(n-1) \qquad y(n) = \sum_{k=-10}^{10} h(k)x(n-k)$$

1.1.3.5. Stable vs. Unstable of Systems. Definitions

An arbitrary relaxed system is said to be ***bounded input - bounded output (BIBO) stable*** if and only if every bounded input produces the bounded output. It means, that there exist some finite numbers say M_x and M_y, such that

$$|x(n)| \leq M_x < \infty \Rightarrow |y(n)| \leq M_y < \infty$$

for all *n*. If for some bounded input sequence *x(n)* , the output *y(n)* is unbounded (infinite), the system is classified as ***unstable***.

Examples:

The stable systems:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) \qquad y(n) = x(n^2) + 3x(n-k)$$

The unstable system:

$$y(n) = 3^n x^3(n-1)$$

1.1.3.6. Recursive vs. Non-recursive Systems. Definitions

A system whose output *y(n)* at time *n* depends on any number of the past outputs values (e.g. *y(n-1)*, *y(n-2)*, ...), is called a ***recursive system***. Then, the output of a causal recursive system can be expressed in general as

$$y(n) = F\left[y(n-1),\, y(n-2),\, \dots,\, y(n-N),\, x(n),\, x(n-1),\, \dots, x(n-M)\right]$$

where *F[.]* is some arbitrary function. In contrast, if *y(n)* at time *n* depends only on the present and past inputs

$$y(n) = F\left[x(n), x(n-1),\, \dots, x(n-M)\right]$$

then such a system is called ***nonrecursive***.

Examples:

The nonrecursive system:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

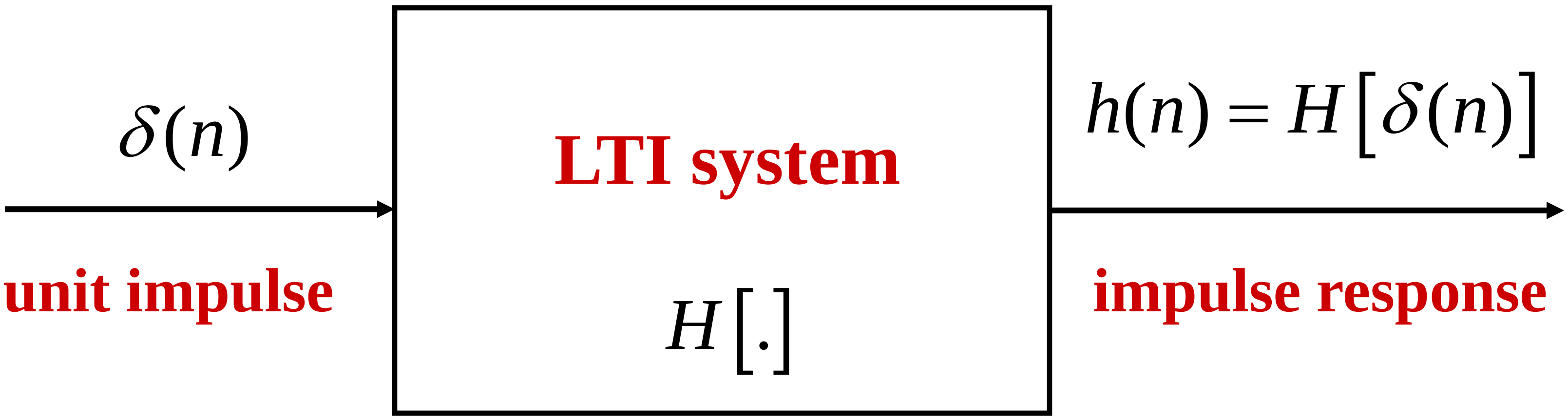
The recursive system:

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{N} a(k)y(n-k)$$

1.2. Linear-Discrete Time Time-Invariant Systems (LTI Systems)

1.2.1. Time-Domain Representation

1.2.1.1 Impulse Response and Convolution

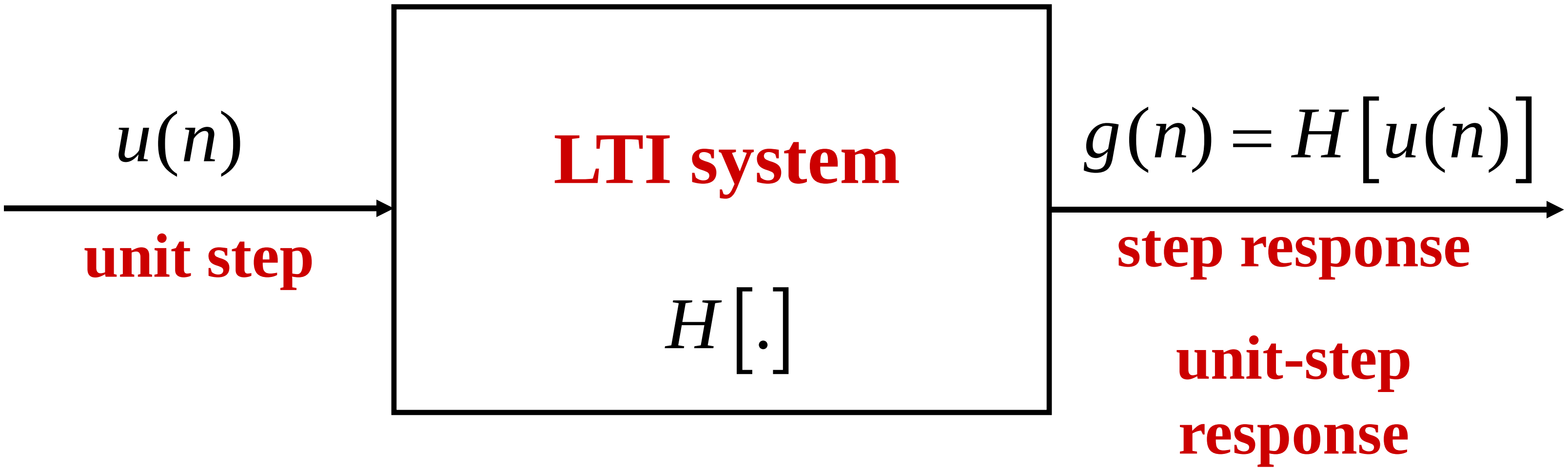


LTI system description by **convolution** (convolution sum):

$$y(n)=\sum_{k=-\infty}^{\infty}h(k)x(n-k)=\sum_{k=-\infty}^{\infty}x(k)h(n-k)=h(n)*x(n)=x(n)*h(n)$$

Viewed mathematically, the convolution operation satisfies the commutative law.

1.2.1.2. Step Response



$$g(n) = \sum_{k=-\infty}^{\infty} h(k)u(n-k) = \sum_{k=-\infty}^{n} h(k)$$

These expressions relate the impulse response to the step response of the system.

1.2.2. Impulse Response Property and Classification of LTI Systems

1.2.2.1. Causal LTI Systems

A relaxed LTI system is ***causal*** if and only if its impulse response is zero for negative values of n , i.e.

$$h(n) = 0 \quad for \quad n < 0$$

Then, the two equivalent forms of the convolution formula can be obtained for the causal LTI system:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{n} x(k)h(n-k)$$

1.2.2.2. Stable LTI Systems

A LTI system is ***stable*** if its impulse response is absolutely summable, i.e.

$$\sum_{k=-\infty}^{\infty} \left|h(k)\right|^2 < \infty$$

1.2.2.3. Finite Impulse Response (FIR) LTI Systems and Infinite Impulse Response (IIR) LTI Systems

Causal **FIR** LTI systems: $$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

IIR LTI systems: $$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

1.2.2.4. Recursive and Nonrecursive LTI Systems

Causal ***nonrecursive*** LTI: $$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

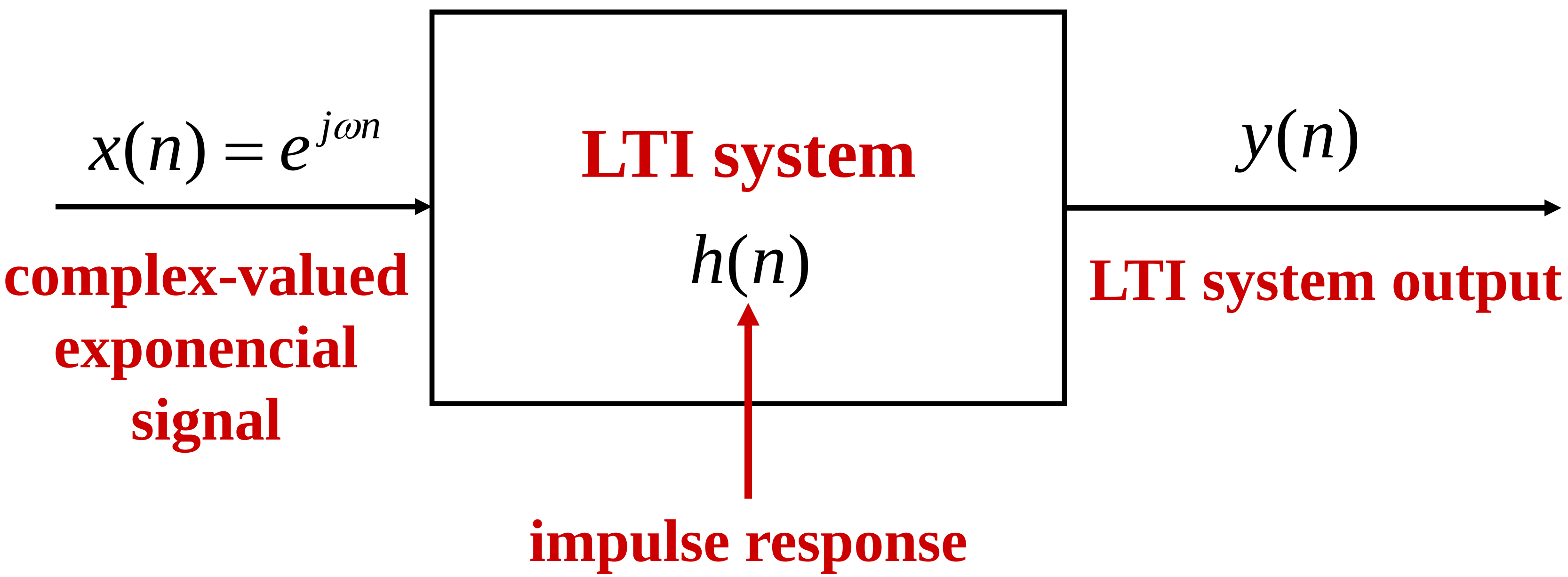
Causal ***recursive*** LTI:

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{M} a(k)y(n-k)$$

LTI systems:

characterized by ***constant-coefficient difference equations***

1.3. Frequency-Domain Representation of Discrete Signals and LTI Systems



$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

LTI system output:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)} =$$

$$= \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}e^{j\omega n} = e^{j\omega n}\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

$$y(n) = e^{j\omega n}H(e^{j\omega})$$

Frequency response: $$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\phi(\omega)}$$

$$H(e^{j\omega}) = \mathrm{Re}\left[H(e^{j\omega})\right] + j\,\mathrm{Im}\left[H(e^{j\omega})\right]$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)\cos\omega k + j\left[-\sum_{k=-\infty}^{\infty} h(k)\sin\omega k\right]$$

$$\mathrm{Re}\left[H(e^{j\omega})\right] = \sum_{k=-\infty}^{\infty} h(k)\cos\omega k$$

$$\mathrm{Im}\left[H(e^{j\omega})\right] = -\sum_{k=-\infty}^{\infty} h(k)\sin\omega k$$

Magnitude response:

$$\left|H(e^{j\omega})\right| = \sqrt{\mathrm{Re}\left[H(e^{j\omega})\right]^2 + \mathrm{Im}\left[H(e^{j\omega})\right]^2}$$

Phase response:

$$\phi(\omega) = \arg\left[H(e^{j\omega})\right] = arctg\,\frac{\mathrm{Im}\left[H(e^{j\omega})\right]}{\mathrm{Re}\left[H(e^{j\omega})\right]}$$

Group delay function:

$$\tau(\omega) = -\frac{d\phi(\omega)}{d\omega}$$

1.3.1. Comments on relationship between the impulse response and frequency response

The important property of ***the frequency response***

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} = \sum_{k=-\infty}^{\infty} h(k)e^{-j[\omega+2l\pi]} = H(e^{j[\omega+2l\pi]})$$

is fact that this function ***is periodic with period*** 2π.

In fact, we may view the previous expression as the exponential Fourier series expansion for $H(e^{j\omega})$, with *h(k)* as the Fourier series coefficients. Consequently, the unit impulse response *h(k)* is related to $H(e^{j\omega})$ through the integral expression

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

1.3.2. Comments on symmetry properties

For LTI systems with real-valued impulse response, the magnitude response, phase responses, the real component of and the imaginary component of $H(e^{j\omega})$ possess these symmetry properties:

The real component: ***even function*** of ω periodic with period 2π

$$\mathrm{Re}\left[H(e^{-j\omega})\right] = \mathrm{Re}\left[H(e^{j\omega})\right]$$

The imaginary component: ***odd function*** of ω periodic with period 2π

$$\mathrm{Im}\left[H(e^{-j\omega})\right] = -\mathrm{Im}\left[H(e^{j\omega})\right]$$

The magnitude response: even function of ω periodic with period 2π

$$\left|H(e^{j\omega})\right| = \left|H(e^{-j\omega})\right|$$

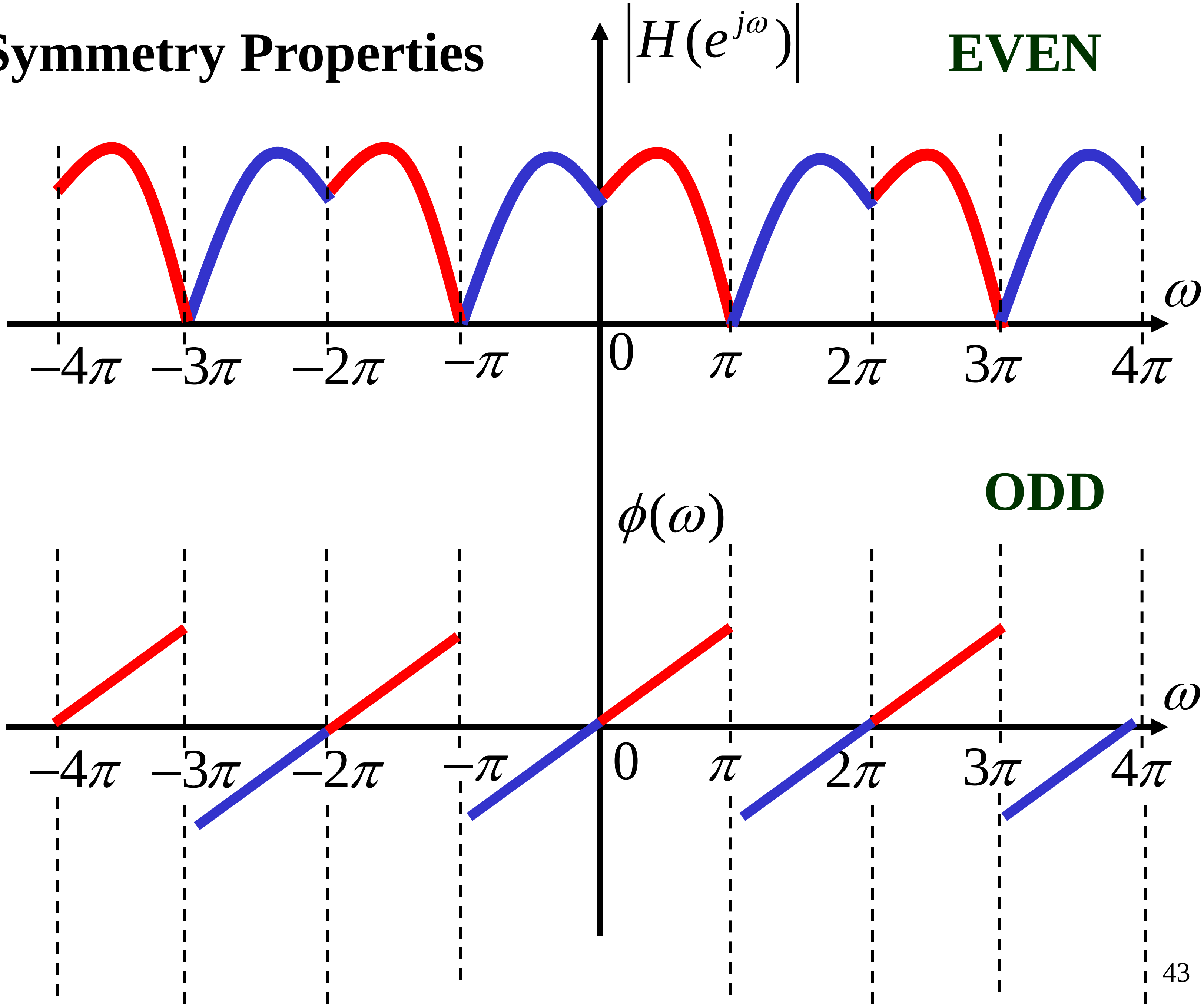
The phase response: odd function of ω periodic with period 2π

$$\arg\left[H(e^{-j\omega})\right] = -\arg\left[H(e^{j\omega})\right]$$

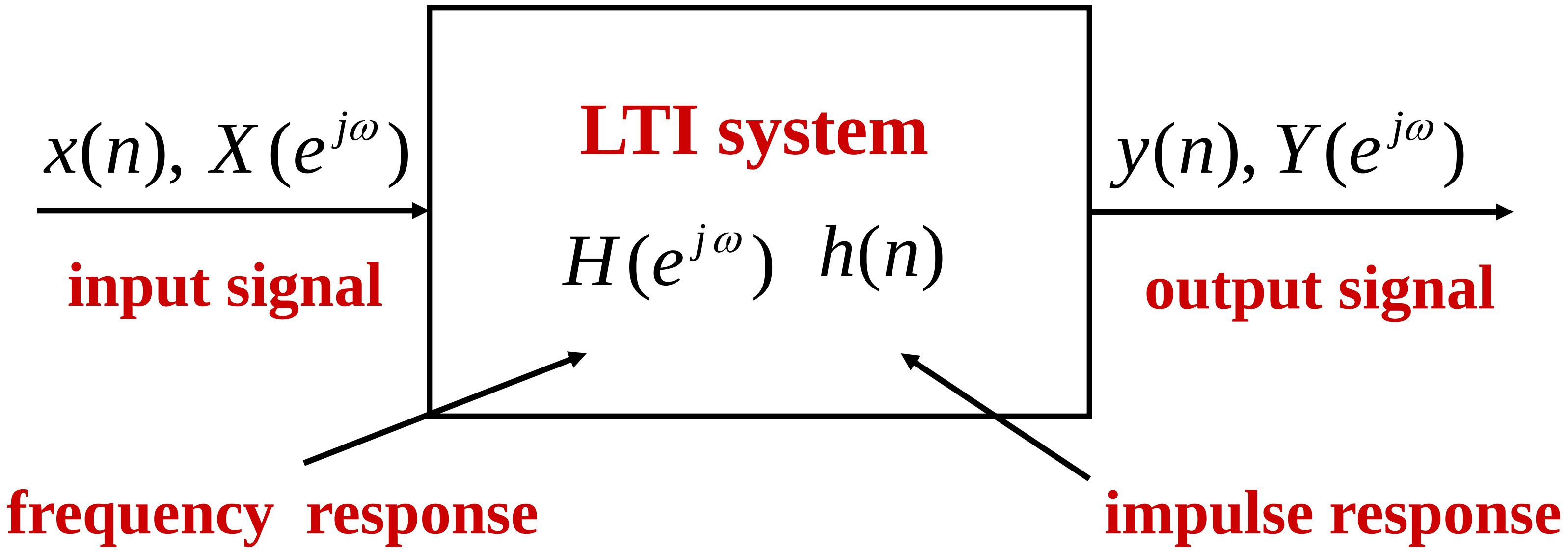
Consequence:

If we known $\left|H(e^{j\omega})\right|$ and $\phi(\omega)$ for $0 \le \omega \le \pi$, we can describe these functions (i.e. also $H(e^{j\omega})$) for all values of ω .

Symmetry Properties



1.3.3. Comments on Fourier Transform of Discrete Signals and Frequency-Domain Description of LTI Systems



The input signal *x(n)* and the spectrum of *x(n)*:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k} \qquad x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

The output signal *y(n)* and the spectrum of *y(n)*:

$$Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} y(k)e^{-j\omega k} \qquad y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega n}d\omega$$

The impulse response *h(n)* and the spectrum of *h(n)*:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \qquad h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

Frequency-domain description of LTI system:

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

1.3.4. Comments on Normalized Frequency

It is often desirable to express the frequency response of an LTI system in terms of units of frequency that involve sampling interval *T*. In this case, the expressions:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \qquad h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

are modified to the form:

$$H(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} h(kT)e^{-j\omega kT}$$

$$h(nT) = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} H(e^{j\omega T})e^{j\omega nT}d\omega$$

$H(e^{j\omega T})$ is periodic with period $2\pi/T = 2\pi F$, where F is sampling frequency.

Solution: **normalized frequency approach**: $F/2 \rightarrow \pi$

Example:

$F = 100kHz \quad F/2 = 50kHz \quad 50kHz \rightarrow \pi$

$$f_1 = 20kHz \qquad \omega_1 = \frac{20x10^3}{50x10^3}\pi = \frac{2\pi}{5} = 0.4\pi$$

$$f_2 = 25kHz \qquad \omega_2 = \frac{25x10^3}{50x10^3}\pi = \frac{\pi}{2} = 0.5\pi$$